Economics Letters 161 (2017) 130–132

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

A concave log-like transformation allowing non-positive values

Martin Ravallion [1]

Department of Economics, Georgetown University, Washington, DC, 20057, USA

HIGHLIGHTS

- The log transformation is widely used in economics but cannot be applied to non-positive data.
- This severely limits applications, including in measuring income and wealth inequality.
- A hybrid of the hyperbolic sine and its inverse is proposed to address this problem.
- Applications are provided, including in an Addendum.

ARTICLE INFO

Article history:
Received 11 April 2017
Received in revised form 19 August 2017
Accepted 18 September 2017
Available online 7 October 2017

JEL classification:
D31
I32

Keywords:
Negative income
Net wealth
Concavity
Hyperbolic sine

ABSTRACT

A hybrid of the hyperbolic sine and its inverse is proposed for applications that require a concave log-like transformation of non-positive data, such as in measuring income and wealth inequality.

1. Introduction

The log transformation is often used to help stabilize the variance of a data series and in measuring growth rates. This paper is concerned with another set of applications in which the log transformation is used in measuring social welfare, inequality and poverty using empirical distributions of income or wealth. The problem then arises that these variables can take non-positive values. Household income measured from surveys can be zero in some period, and even negative given that it should include profit from own-enterprises. Similarly, there can be non-positive values of net wealth (assets less liabilities). A response found in practice is to treat the troublesome non-positive values as missing. This is clearly unsatisfactory since they are not in fact missing (and certainly not randomly). Another common response is to set the negatives to zero, in the belief that they are purely transient. While there may well be a large transient element that does not imply that the negatives should be set to zero, while keeping untouched the (undoubtedly present) transient positives. One still wants to measure poverty and inequality in cross-sectional surveys even acknowledging that there are transient components.

Non-positive values are in fact quite common for economic variables. For about 400 of the 700 income surveys used in the World Bank's PovcalNet data site for global poverty and inequality measurement have non-positive values for household income.[2] Similarly, non-positive values of net wealth (assets less liabilities) are common. The US Census Bureau data indicate that 18% of US households in 2011 had zero or negative net wealth. If one is studying the net wealth of young American households (headed by someone 18–24 years of age) then one is not able to use the log transformation for about one-third of those households![3]

E-mail address: mr1185@georgetown.edu.

1 For useful comments or discussions on this paper the author thanks Francois Bourguignon, Christopher Camenares, Dan Cao, Denis Cogneau, Shaohua Chen, Garance Genicot, Madhulika Khanna, Franco Peracchi and the journal's editor and reviewers.

2 358 have negative incomes, and 48 have zeros (The author thanks Shaohua Chen for providing this information.).

3 This is shown in the statistical addendum to this paper.

https://doi.org/10.1016/j.econlet.2017.09.019
0165-1765/

This paper proposes a transformation that can accommodate negative values but has properties suitable for measuring inequality and poverty, unlike past practices in the literature. A statistical addendum provides illustrations of the main points using real data.

2. Applications of log and log-like transformations

Johnson (1949) introduced the inverse hyperbolic sine (IHS) transformation as a means of making non-normally distributed data more normally distributed—essentially to "stabilize the variance". For real $y \in [y^{\min}, y^{\max}]$, the IHS transformation is $\sinh^{-1}(y) \equiv \ln\varphi(y)$ where $\varphi(y) \equiv y + (y^2+1)^{0.5} > 0$. For $y > 0$, the transformation is "log-like" in that it becomes more like $\ln 2y$ as y rises $\left(\lim_{y\to\infty}[\ln(\varphi(y)/2) - \ln y] = 0\right)$. (It is also sign-preserving, returning the sign of y).

The IHS has recently become popular in economics.[4] An EconLit text search for "inverse hyperbolic sine" delivers 150 papers, with half of them since 2010. With reference to household net-wealth data, a journal editor writes that: "I find myself this morning, once again, writing a revise-and-resubmit letter along the lines of 'and re-do the estimation using an inverse hyperbolic sine transformation'" (Woolley 2011).

In addition to stabilizing the variance of a series of positive values, the log transformation has served other purposes in economics. One is in measuring growth rates. Practices vary, but when measuring growth rates in a series that has non-positive values an obvious solution is $g(y) \equiv -I\ln(-y) + (1-I)\ln y$ where I takes the value unity if $y \le 0$ and zero otherwise. Then $dg(y) = dy/|y|$.[5]

The log transformation has also been popular in measuring individual welfare as a function of (positive) income or wealth. The function emerges as a special case for inequality measures anchored to explicit social welfare functions, such as the arithmetic sum of welfare, as in the measures proposed by Dalton (1920) and Atkinson (1970). Another example is the Mean Log Deviation (*MLD*) (one of the measures proposed by Theil, 1967): $MLD = \sum_{i=1}^{n} \ln(\bar{y}/y_i)$ for n positive incomes with mean $\bar{y}$. (*MLD* is strictly positive, given that $\ln y$ is concave, but is unbounded above.) While the *MLD* is known to have a number of desirable properties, including decomposability using population weights (Bourguignon, 1979), the fact that it requires incomes to be logged has limited its applications. The same problem arises for the Watts (1968) index of poverty, which is known to have more desirable theoretical properties than other measures (Zheng, 1993) yet is rarely used. Indeed, none of these measures can be applied when some of the data on incomes or wealth take non-positive values.

In the context of individual welfare measurement, one can identify two theoretically desirable properties of a function $h(y)$ applied to income or wealth y:

(a) Continuity and monotonicity: The function exists for all y in $(-\infty, +\infty)$, is continuous with a continuous and differentiable first derivative, $h'(y) > 0$ for all y.
(b) Concavity: The function is everywhere strictly concave ($h''(y) < 0$ for all y).

The IHS transformation does not satisfy (b), as it is only strictly concave for $y > 0$; specifically:

$$\frac{d^2\sinh^{-1}(y)}{dy^2} = \frac{-y}{(y^2+1)^{1.5}} > (<)\, 0 \text{ as } y < (>)\, 0.$$

Thus if one switches from the log to the IHS then the modified inequality measure violates the Pigou–Dalton principle. The latter should presumably hold over the entire domain, as an implication of the assumption of declining marginal utility. As an ethical judgement, if one penalizes inequality among those with positive net wealth (say) this would surely also hold when one allows for negative net wealth.

Thus, on using the IHS transformation, there will always be mean-preserving transfers in which the donor is worse off than the recipient that would decrease measured inequality. And this does not only hold among those with negative net wealth; given the *S* shape of the IHS transformation there can be transfers from people with negative net wealth to those with positive net wealth that will decrease measured inequality. To give a numerical example, suppose that the distribution of net wealth includes -2 and 1. It is readily verified that taking 0.2 (say) of wealth away from the poorer person, with -2, and giving it to the person with 1 will increase the sum of their IHS-transformed wealth values. This mean-preserving but inequality-increasing transfer will be deemed to be social welfare improving using the IHS transformation.

3. Concave log-like transformation allowing non-positive values

One corrective that already exists in the literature is the "started log", $\ln(y+c)$ where the parameter $c > 0$ is set such that $y + c > 0$ for all y (Tukey, 1977). However, the started log runs into the problem that it is close to linear for large c (the second derivative is $-(y+c)^{-2}$). The Addendum provides an example using income data (for Belgium) with large negatives in which inequality measure based on started logs goes to virtually zero (*MLD* = 0.01). This is clearly deceptive. What might be used instead of the started log?

The proposal here is to use the ordinary hyperbolic sine transformation for negative y, and only use the inverse function for positive y. (Notice that one cannot switch to $\ln y$ for $y > 0$ because this creates a discontinuity at $y = 0$.) The hyperbolic sine transformation is $\sinh(y) \equiv 0.5\left(e^y - e^{-y}\right)$ with $\sinh'(y) = 0.5\left(e^y + e^{-y}\right)$ (the hyperbolic cosine) and $\sinh''(y) = \sinh(y) > (<)\, 0$ as $y > (<)\, 0$. The idea is simple: switching to the (normal) hyperbolic sine function for $y < 0$ naturally reverses the troubling convexity of its inverse function. The latter still plays a role for $y > 0$, such that continuity is assured.

Thus the proposed transformation is:

$$h(y) \equiv I\sinh(y) + (1-I)\sinh^{-1}(y) - \ln 2. \quad (1)$$

We have $\sinh(0) = \sinh^{-1}(0) = 0$, so $h(y)$ is continuous at $y = 0$, as elsewhere, with $h(y) > (<)\, 0$ as $y > (<)\, 0$. (Continuity assures that $h(y)$ can be inverted.) The transformation $h(y)$ also has a continuous first derivative; the first derivatives of both $\sinh(y)$ and $\sinh^{-1}(y)$ approach unity as y approaches zero from the negative and positive sides respectively. (The term $-\ln 2$ in (1) assures that $\lim_{y\to\infty}[h(y) - \ln y] = 0$.) Note that $h(y)$ is also sign-preserving, i.e., $h(0) = 0$ and $h(y) > (<)\, y$ as $y > (<)\, 0$.

Naturally, a strictly concave transformation that exists for negative values will "spread out" these negative values, rather than compress them, as in the IHS transformation. Thus the transformation in (1) can produce large negatives; intuitively, the transformation "de-stabilizes" the variance. This is an inevitable implication of using a transformation that is concave over the entire real line.

The proposed transformation can be parametrized as:

$$h(y;\alpha) \equiv I\sinh(\alpha y) + (1-I)\sinh^{-1}(\alpha y) - \ln 2\alpha \;(\alpha > 0). \quad (2)$$

Note that $\partial h(0;\alpha)/\partial y = \alpha$, i.e., the function flattens out near the origin as one lowers α. When the transformation in (2) is

[4] Other log-like transformations have been proposed including the generalized log (Durbin et al., 2002), the log modulus (John and Draper, 1980), the log-linear hybrid (Rocke and Durbin, 2003) and the hyperlog (Bagwell, 2005). The following observations also apply to these functions.

[5] Like the proportionate change, $dg(y)$ does not exist at $y = 0$.

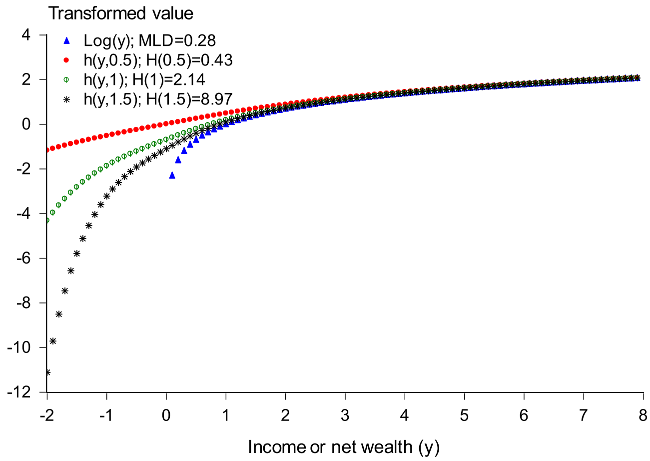

Fig. 1. h-transformations for different parameter values.

used in econometric modelling, α is estimable by the method of maximum likelihood, similarly to applications using the IHS (following Burbidge et al., 1988); the addendum gives an empirical example in modelling wealth dynamics. In other applications (such as measuring inequality and poverty) α must be set by the analyst.

Fig. 1 plots the transformation in (2) applied to numbers in the interval $(-2, 8)$. (These values would be typical of the low-middle incomes in rich countries in units of $1,000 per person per year, or incomes in developing countries in units of $'s per person per day.) We see that $h(y; \alpha)$ approaches $\ln y$ quickly. For $\alpha = 1.5$, the hyperbolic sine yields a deep curvature, with large negatives at the extremes. Inevitably, as one gets close to zero the transformed values differ markedly to the log.

As an intuitive summary statistic for inequality, Fig. 1 also gives:

$$H(\alpha) \equiv h(\bar{y}; \alpha) - \frac{1}{n}\sum_{i=1}^{n} h(y_i; \alpha) \left(\text{where } \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i\right). \quad (3)$$

This will recognized be as a modified *MLD* in which $\ln y$ is replaced by $h(y; \alpha)$. So it is of interest to compare $H(\alpha)$ to *MLD*, which is 0.28 for these data. We see that $H(\alpha)$ is substantially higher. This suggests that how one deals with negative net wealth or income data can make a big difference to measures of inequality.

4. Conclusion

The *S*-shape on the real line of the popular inverse hyperbolic sine transformation makes it problematic in some economic applications. For example, measured inequality can respond perversely to changes in the distribution of income or wealth distribution. When a continuous concave log-like transformation is called for with data that include non-positive values the composite of the primal and inverse hyperbolic sign functions proposed here should be considered.

References

Atkinson, Anthony B., 1970. On the measurement of inequality. J. Econ. Theory 2, 244–263.
Bagwell, C. Bruce, 2005. HyperLog—A flexible log-like transform for negative, zero and positive valued data. Cytometry Part A 64A, 34–42.
Bourguignon, François, 1979. Decomposable income inequality measures. Econometrica 47, 901–920.
Burbidge, John B., Magee, Lonnie, Leslie Robb, A., 1988. Alternative transformations to handle extreme values of the dependent variable. J. Am. Stat. Assoc. 83 (401), 123–127.
Dalton, Hugh, 1920. The measurement of the inequality of incomes. Econ. J. 30 (119), 348–361.
Durbin, Blythe, Hardin, J.S., Hawkins, D.M., Rocke, D.M., 2002. A variance-stabilizing transformation for Gene-expression microarray data. Bioinformatics 18, S105–S110.
John, J.A., Draper, N.R., 1980. An alternative family of transformations. Appl. Stat. 29 (2), 190–197.
Johnson, N.L., 1949. Systems of frequency curves generated by methods of translation. Biometrika 36, 149–176.
Rocke, D.M., Durbin, B., 2003. Approximate variance-stabilizing transformations for gene-expression microarray data. Bioinformatics 19, 966–972.
Theil, Henri, 1967. Economics and Information Theory. North-Holland, Amsterdam.
Tukey, John W., 1977. Exploratory Data Analysis. Addison-Wesley, Reading, MA.
Watts, Harold W., 1968. An economic definition of poverty. In: Moynihan, Daniel P. (Ed.), On Understanding Poverty. Basic Books, New York.
Woolley, Frances, 2011, A Rant on Inverse Hyperbolic Sine Transformations, Worthwhile Canadian Initiative.
Zheng, Buhong, 1993. Axiomatic characterization of the watts index. Econ. Lett. 42, 81–86.